Research Article

All Congruence Modular Symmetric and Near-Symmetric Algebras

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Abstract

For a unary operation f on a finite set A, let denote $\lambda(f)$ the least non-negative integer with $\operatorname{Im} f^{\lambda(f)} = \operatorname{Im} f^{\lambda(f)+1}$ which is called the pre-period of f. K. Denecke and S. L. Wismath have characterized all operations f on A with $\lambda(f) = |A| - 1$ and prove that $\lambda(f) = |A| - 1$ if and only if there exists a $d \in A$ such that $A = \{d, f(d), f^2(d), \ldots, f^{|A|-1}(d)\}$ where $f^{|A|-1}(d) = f^{|A|}(d)$. C. Ratanaprasert and K. Denecke have characterized all operations f on A with $\lambda(f) = |A| - 2$ for all $|A| \geq 3$; and have characterized all equivalence relations on A which are invariant under f with these long pre-periods.

In the paper, we study finite unary algebras $\overline{A} = (A; f)$ with $\lambda(f) \in \{0, 1\}$ for $|A| \geq 3$ which are called symmetric algebras and near-symmetric algebras, respectively. We characterize all operations f whose $\overline{A}$ is congruence modular. We prove that a symmetric algebra $\overline{A}$ is congruence modular if and only if the lattice $Con\overline{A}$ of all congruence relations is either a product of chains or a linear sum of a product of chains with one element top or a M_3-head lattice; and a near-symmetric algebra $\overline{A}$ is congruence modular if and only if $Con\overline{A}$ is one of the followings:

$$\underline{2} \times P,\ \underline{2} \times (P \oplus \underline{1}),\ \underline{2} \times L,\ M_3 \times P,\ M_3 \times (P \oplus \underline{1}),\ \text{or}\ M_3 \times L$$

where P denote a product of chains and L is a M_3- head lattice.

Key Words: Monounary algebra; Congruence distributive; Congruence modular

Introduction

Monounary algebras are algabras with one unary fundamental operation. Mono-unary and partial monounary algebras play a significant role in the study of algebraic structures. Moreover, there exists a close connection between monounary algebras and some types of automata. The advantage of monounary algebras is their relatively simple visualization. They can be represented by a graph, which is always planar, hence easy to draw. Unary algebras were first intensively investigated by B. Johnson about 40 years ago and were investigated mainly by D. J. Studenovska (1982, 1983). The problem of describing the lattices which are isomorphic to congruence lattices of monounary algebras is still open.

Let A be a finite set and denote by $|A| \geq 2$ the cardinality of A. For a unary operation f on A, let $\operatorname{Im} f = \{f(x) | x \in A\}$ be the image of f and let $\lambda(f)$ be the least non-negative integer m such that $\operatorname{Im} f^m = \operatorname{Im} f^{m+1}$. The number $\lambda(f)$ is called the pre-period of f, sometimes also the *stabilizer* of f. K. Denecke and S.L. Wismath (2002) have proved that $\lambda(f) = |A| - 1$ if and only if there exists a $d \in A$ such that $A = \{d, f(d), f^2(d), \ldots, f^{|A|-1}(d)\}$ which shows a characterization of all longest pre-periods f.

It is well-known that the congruence lattice of an algebra is uniquely determined by the unary polynomial operations of the algebra. C. Ratanaprasert and K. Denecke (2008) have characterized all unary operations f on a finite set A with $\lambda(f) = |A| - 2$ for $|A| \geq 3$ and they also have characterized all equivalence relations on A which are invariant under f with $\lambda(f) = |A| - 1$ for $|A| \geq 2$ and $\lambda(f) = |A| - 2$ for $|A| \geq 3$. These answer the above open problem for some of monounary algebras. Besides, the results convince us that the pre-period of unary functions defined on a finite set will be a kind of notions for classifications of finite algebras. At the beginning of the eighties, R. McKenzie and D. Hobby (1998) developed a new theory, called "Tame Congruence Theory" which offers a structure theory for finite algebras. If $|\operatorname{Im} f| = |A|$ or $|\operatorname{Im} f| = 1$, then $\overline{A} = (A; f)$ is called a permutation algebra; that is, $\lambda(f) \in \{0, 1\}$ and f is of short pre-periods. Permutation algebras play an important role in tame congruence theory.

In this paper, we are interested in formulating a characterization of all unary operations defined on a finite set A with short pre-periods. We prove necessary and sufficient conditions of f whose permutation algebras are congruence modular and then characterize all lattices which are isomorphic to the congruence lattice of modular permutation algebras.

All lattices which are the congruence lattices of modular symmetric algebras.

In this section, we assume that A is a finite set and f is a unary operation on A with $\lambda(f) = 0$. Note that $\lambda(f) = 0$ if and only if f is a permutation on A. In the theory of groups, the group of all permutations on a nonvoid set A is known as a symmetric group and it is well known that every permutation can be decomposed into simple parts called cycles. We call the monounary algebra $\overline{A}=(A;f)$ in this case, a symmetric algebra. If A is a singleton or a two-elements set, the lattice $Con\overline{A}$ of all congruence relations on $\overline{A}$ is also a singleton set $\{\Delta_A\}$ or a two-elements chain $\{\Delta_A, A\times A\}$, respectively; so, $\overline{A}$ is congruence-distributive. We are interested in the case $|A|\geq 2$. We first consider necessary conditions for f whose $\overline{A}$ is congruence-distributive.

Recall that the notation $\theta(B)$ stands for the least congruence on an algebra $\overline{A}$ which contains a subset B of A.

Proposition 1. If $\overline{A}$ is a congruence distributive symmetric algebra with $|A|\geq 3$ then either f is a cycle having at most one fixed point or, f has no fixed points and f is a product of two disjoint cycles whose lengths are relatively prime.

Proof. Suppose that $f=\alpha_1\alpha_2\ldots\alpha_r$ for $r\geq 3$ and α_i and α_j are disjoint cycles for all $1\leq i\neq j\leq r$. Let $\sigma=(123)$ and denote by B_i, the set of all elements in the cycle α_i for all $1\leq i\leq r$. Since $B_i\cap B_j$ is empty and $f(x)\in B_i$ whenever $x\in B_i$ for all $1\leq i\leq r$, the relations $\theta_j:=\Delta_A\cup \left\{(x,y)\,|\,\{x,y\}\subseteq B_j\cup B_{\sigma(j)} \text{ or } \{x,y\}\subseteq B_{\sigma^2(j)}\right\}$ is invariant under f and $\omega:=\theta_j\wedge\theta_{\sigma(j)}=\Delta_A\cup\bigcup_{k=1}^{3}\{(x,y)\,|\,x,y\in B_k\}$ for all $j\in\{1,2,3\}$. Also, the congruence $\theta=\theta(B_1\cup B_2\cup B_3)$ contains θ_j for all $j\in\{1,2,3\}$; and if $(a,b)\in\theta$ then $\{a,b\}\subseteq B_j\cup B_{\sigma(j)}$ for some $j\in\{1,2,3\}$; hence, $(a,b)\in\theta_j\vee\theta_{\sigma(j)}$; so, $\theta_j\vee\theta_{\sigma(j)}=\theta$ for all $j\in\{1,2,3\}$. Therefore, $Con\overline{A}$ is not distributive since it contains a M_3-sublattice $\{\theta_1,\theta_2,\theta_3,\omega,\theta\}$. □

We note that if $|A|=3$ and $Con\overline{A}$ is not distributive then $Con\overline{A}$ is the modular lattice M_3.

Remark 1. Let $(B;f^B)$ be a subalgebra of $\overline{A}$. We will denote the restriction $\theta|_B$ of $\theta\subseteq A\times A$ on B by θ_B. And if $\theta\subseteq B\times B$, we will denote the relation $\theta\cup\{(x,x)\,|\,x\in A\}$ by θ^A. Then $\theta_B\in Con(B;f^B)$ for all $\theta\in Con\overline{A}$ and $\theta^A\in Con\overline{A}$ for all $\theta\in Con(B;f^B)$.

Remark 2. Let $(af(a)\ldots f^{p-1}(a))$ and $(bf(b)\ldots f^{q-1}(b))$ be cycles in the product of f for some positive integers p and q and $\theta\in Con\overline{A}$.

(i) If $(a,f^r(a))\in\theta$ for some $0<r\leq p-1$, then $(a,f^k(a))\in\theta$ for all non-negative integer k.

(ii) If $(a,f^r(a))\in\theta$ for some integer $0<r\leq p-1$ which is not a factor of p then $\{a,f(a),\ldots,f^{p-1}(a)\}$ is contained in a block of the quotient algebra A/θ.

(iii) If $(p,q)=1$ and $(a,b)\in\theta$, then $\{a,f(a),\ldots,f^{p-1}(a),b,f(b),\ldots,f^{q-1}(b)\}$ is contained in a block of A/θ.

Recall that a linear sum of an ordered set P with a one-element chain $\underline{1}$ is an ordered set $P\oplus\underline{1}$ which can represents P with a new top element added.

Proposition 2. If f satisfies the conditions of Proposition 1 then $Con\overline{A}$ is either a product of chains or a linear sum of a product of chains with a one-element chain.

Proof. Let $|A| = n$ and f be a cycle having no fixed point and $a \in A$. Let denote $\downarrow n$ the lattice of all factors of n ordered by the division of integers. For each $m \in \downarrow n$, let $f^{[j]_m}(a) = \{ f^s(a) \mid s \equiv j \pmod{m}\}$. Then $\wp_m := \{ f^{[j]_m}(a) \mid j = 0,1,2,\ldots,m-1\}$ is a partition of A which corresponds to the congruence θ_m modulo m restriction to A; that is, $\theta_m = \{(x,y) \mid x,y \in f^{[j]_m}(a) \text{ for } j = 0,1,2,\ldots,m-1\}$. Hence, the map $\alpha : \downarrow n \to Con\overline{A}$ defined by $\alpha(m) = \theta_m$ for all $m \in \downarrow n$ is clearly an order-isomorphism; so, $Con\overline{A}$ is dually isomorphic to $\downarrow n$ which is a product of chains.

Next, assume that $f = \alpha_1\alpha_2$ where α_1 and α_2 are disjoint cycles whose lengths are relatively prime whenever both of them are of lengths more than one. Then, $f|_{B_i}$ is a cycle on the set B_i of all elements in the cycle α_i for $i \in \{1,2\}$. Hence, $Con(B_i; f|_{B_i})$ is a product of chains for $i \in \{1,2\}$. Since $\{B_1, B_2\}$ is a the partition on A, $\theta_1^A \cup \theta_2^A$ is a congruence on $\overline{A}$ for all $\theta_i \in Con(B_i; f|_{B_i})$ and $i \in \{1,2\}$; hence, $Con\overline{A}$ is a sublattice of the power set of $A \times A$,; so, the map $\beta : (\theta_1, \theta_2) \to \theta_1^A \vee \theta_2^A$ is an order embedding from $Con(B_1; f|_{B_1}) \times Con(B_2; f|_{B_2})$ into $Con\overline{A}$.

Let $\theta_i \in Con(B_i; f|_{B_i})$ for $i \in \{1,2\}$. Then $(x,y) \notin \theta_1^A \vee \theta_2^A$ whenever $x \in B_1$ and $y \in B_2$; so, $\theta_1^A \vee \theta_2^A \neq A \times A$; hence, $A \times A \notin \mathrm{Im}\,\beta$. Now, if $\theta \in Con\overline{A} \setminus \{A \times A\}$ then $\theta_{B_1}^A \cup \theta_{B_2}^A \in \mathrm{Con}\overline{\mathrm{A}}$ where $\theta_{B_1}^A \cup \theta_{B_2}^A \subseteq \theta$. If $(a,b) \in \theta$ with $a \in B_1$ and $b \in B_2$, the result (iii) in Remark 2 implies that $A = B_1 \cup B_2$ is a subset of a block of A/θ since the lengths of α_1 and α_2 are relatively prime; so, $\theta = A \times A$, a contradiction. So, if $(a,b) \in \theta$ then $\{a,b\} \subseteq B_i$ for some $i \in \{1,2\}$; hence, $(a,b) \in \theta_{B_i} \subseteq \theta_{B_1}^A \cup \theta_{B_2}^A$. Thus, $\theta \in \mathrm{Im}\,\beta$. Therefore, $Con\overline{A} \setminus \{A \times A\} = \mathrm{Im}\,\beta \cong Con(B_1; f|_{B_1}) \times Con(B_2; f|_{B_2})$. Hence, $Con\overline{A}$ is a linear sum of a product of chains P and a one-element chain $\underline{1}$. □

We knew that an algebra $\overline{A}$ is congruence-distributive if A is singleton or a two-elements set. In the case $|A| = 3$, if f is identity then $Con\overline{A}$ is modular; and if f is not identity, Proposition 2 implies that $Con\overline{A}$ is distributive. We will consider the case $|A| \geq 4$.

Proposition 3. If a symmetric algebra $\overline{A}$ with $|A| \geq 4$ is congruence-modular, then f is either one of the followings:

(i) f is a cycle having at most two fixed points, or

(ii) f has at most one fixed point and f is a product of two disjoint cycles whose lengths are relatively prime, or

(iii) f has no fixed point and f is a product of three disjoint cycles whose lengths are relatively prime.

Proof. Similar arguing as the proof of Proposition 1. Suppose that f is a product of at least four disjoint cycles $\alpha_1\alpha_2\ldots\alpha_r$ where $r \geq 4$ and all α_i and α_j are disjoint cycles (can be of length 1) for $1 \leq i \neq j \leq r$. Then, we construct the following 3 congruences :

$$\theta_1 = \Delta_A \cup \{(x,y) \mid \{x,y\} \subseteq B_i \cup B_{\sigma(i)} \ \ or \ \ \{x,y\} \subseteq B_{\sigma^2(i)} \ \ or \ \ \{x,y\} \subseteq B_{\sigma^3(i)}\}$$

$$\theta_2 = \Delta_A \cup \{(x,y) \mid \{x,y\} \subseteq B_i \cup B_{\sigma(i)} \ \ or \ \ \{x,y\} \subseteq B_{\sigma^2(i)} \cup B_{\sigma^3(i)}\} \text{ and}$$

$$\theta_3 = \Delta_A \cup \{(x,y) \mid \{x,y\} \subseteq B_i \cup B_{\sigma^2(i)} \ \ or \ \ \{x,y\} \subseteq B_{\sigma(i)} \cup B_{\sigma^3(i)}\}$$

Where $\sigma = (1234)$. One can show that $\theta_1 \subseteq \theta_2$, $\theta_1 \wedge \theta_3 = \theta_2 \wedge \theta_3 = \Delta_A \cup \bigcup_{k=1}^{4}\{(x,y) \mid x,y \in B_k\}$ and $\theta_1 \vee \theta_3 \subseteq \theta_2 \vee \theta_3$. If $(a,b) \in \theta_2 \vee \theta_3$, there are $a = q_0, q_1, \ldots, q_t = b \in A$ such that $(q_k, q_{k+1}) \in \theta_2 \cup \theta_3$ for all $0 \leq k \leq t-1$. We may assume that $(a, q_1) \in \theta_2$. The finiteness of the set $Q = \{a = q_0, q_1, \ldots, q_t = b\}$ implies the existence of the greatest element $q_k \in Q$ such that $(q_k, q_{k+1}) \in \theta_3$ but $(q_{i-1}, q_i) \in \theta_2$ for all $1 \leq i \leq k$; so, $(a, q_k) \in \theta_2$. If $(q_j, q_{j+1}) \in \theta_3$ for each $k \leq j \leq t-1$ then $(q_k, b) \in \theta_3$; hence, $\{a, q_k\} \subseteq B_i \cup B_{\sigma(i)}$ or $\{a, q_k\} \subseteq B_{\sigma^2(i)} \cup B_{\sigma^3(i)}$. If $\{a, q_k\} \subseteq B_i \cup B_{\sigma(i)}$ then $(a, q_k) \in \theta_1$; so, $(a,b) \in \theta_1 \vee \theta_3$; but, if $\{a, q_k\} \subseteq B_{\sigma^2(i)}$ or $\{a, q_k\} \subseteq B_{\sigma^3(i)}$ then $(a, q_k) \in \theta_3$; so, $(a,b) \in \theta_3 \subseteq \theta_1 \vee \theta_3$. We consider the case $a \in B_{\sigma^2(i)}$ and $q_k \in B_{\sigma^3(i)}$. We have $b \in B_{\sigma(i)}$ or $b \in B_{\sigma^3(i)}$; so, $(a,k) \in \theta_3$ for all $k \in B_i$; hence, $(k,l) \in \theta_1$ for all $k \in B_i$ and $l \in B_{\sigma(i)}$; and also, $(k,s) \in \theta_1$ for all $k \in B_i$ and $s \in B_{\sigma(i)}$ which implies that $(s,m) \in \theta_3$ for all $m \in B_{\sigma^3(i)}$ and $s \in B_{\sigma(i)}$; thus, $(s, q_k) \in \theta_3$ for all $s \in B_{\sigma(i)}$. Now, $a \ \theta_3 \ x \ \theta_1 \ s \ \theta_3 \ q_k \ \theta_3 \ b$ implies that $(a,b) \in \theta_1 \vee \theta_3$. If $(q_j, q_{j+1}) \notin \theta_3$ for all $k \leq j \leq t-1$, the same arguing as above and by continuing the process one can prove that $\theta_2 \vee \theta_3 \subseteq \theta_1 \vee \theta_3$. So, $\theta_1 \vee \theta_3 = \theta_2 \vee \theta_3$.

Therefore, $Con\overline{A}$ is not modular since it contains a N_5 – sublattice $\{\theta_1, \theta_2, \theta_3, \theta_1 \wedge \theta_3, \theta_1 \vee \theta_3\}$. □

We note from Proposition 3 that $Con\overline{A}$ is not modular if $|A| \geq 4$ and f is an identity on A.

Proposition 4. If $\overline{A}$ is a symmetric algebra with $|A| \geq 4$ whose f has no fixed points and f is a product of three disjoint cycles all of them are of relatively prime lengths, then $Con\overline{A}$ is modular which is not distributive.

Proof. Let $f = \alpha_1\alpha_2\alpha_3$ satisfy the conditions of the proposition. Then, Proposition 1 implies that $\mathrm{Con}(B_i; f|_{B_i})$ is a product of chains and $Con(B_i \cup B_j; f|_{B_i \cup B_j})$ is a linear sum of a product of chains P with a one-element chain $\underline{1}$ for all $i, j \in \{1,2,3\}$.

If $\theta = \Delta_A$, clearly $\theta_{B_i} = \Delta_{B_i}$ for all $i \in \{1,2,3\}$. Let $\theta \in Con\overline{A} \setminus \{\Delta_A, A \times A\}$ and assume that $\theta \neq \bigcup_{i=1}^{3} \theta_{B_i}$. Then $\theta_{B_i \cup B_j} \cup \theta_{B_k} \subseteq \theta$ for all $i \in \{1,2,3\}$. If $(x,y) \in \theta$ then $\{x,y\} \not\subset B_i$ for all $i \in \{1,2,3\}$ since $\theta \neq \bigcup_{i=1}^{3} \theta_{B_i}$; so, there are $1 \leq i \neq j \leq 3$ such that $x \in B_i$ and $y \in B_j$; hence, $(x,y) \in \theta_{B_i \cup B_j} \cup \theta_{B_k}$ where $\{i,j,k\} = \{1,2,3\}$. Therefore, $\theta = \bigcup_{i=1}^{3} \theta_{B_i}$. or $\theta = \theta_{B_i \cup B_j} \cup \theta_{B_k}$ where $\{i,j,k\} = \{1,2,3\}$.

Next, let $\beta : (\theta, \phi) \mapsto \overline{\theta} \vee \overline{\phi}$ where $\overline{\theta} = \theta \cup \theta_{B_k}$ and $\overline{\phi} = \phi \cup \Delta_{B_i \cup B_j}$ for all $\theta \in Con(B_i \cup B_j; f|_{B_i \cup B_j})$ and $\phi \in Con(B_k; f|_{B_k})$. If $\theta_t \in Con(B_i \cup B_j; f|_{B_i \cup B_j})$ and $\phi_t \in Con(B_k; f|_{B_k})$ then $\overline{\theta_t} \cup \overline{\phi_t} = \overline{\theta_t} \vee \overline{\phi_t}$ for $t \in \{1,2\}$. Thus, β is an order-embedding since

$$(\theta_1, \phi_1) \subseteq (\theta_2, \phi_2) \Leftrightarrow \theta_1 \subseteq \theta_2 \text{ and } \phi_1 \subseteq \phi_2 \Leftrightarrow \overline{\theta_1} \cup \overline{\phi_1} \subseteq \overline{\theta_2} \cup \overline{\phi_2}$$
$$\Leftrightarrow \overline{\theta_1} \vee \overline{\phi_1} \subseteq \overline{\theta_2} \vee \overline{\phi_2} \Leftrightarrow \beta(\theta_1, \phi_1) \subseteq \beta(\theta_2, \phi_2)$$

For each $i \in \{1,2,3\}$, let C_i be a sublattice of $Con\overline{A}$ which is isomorphic to $Con(B_i \cup B_{\sigma(i)}; f|_{B_i \cup B_{\sigma(i)}}) \times Con(B_{\sigma^2(i)}; f|_{B_{\sigma^2(i)}})$ where $\{i,j,k\} = \{1,2,3\}$ and let m_i be the greatest element of C_i. We will show that m_1, m_2 and m_3 are the only co-atoms of $Con\overline{A}$. First, $m_i \neq A \times A$ for $i \in \{1,2,3\}$ since $(x,y) \notin m_i$ for all i i and all $x \in B_i$ and $y \in B_{\sigma^2(i)}$. Secondly, let $i \in \{1,2,3\}$ and $m_i \subset \theta \subseteq A \times A$. Then, there exist $(a,b) \in \theta$ and $(a,b) \notin m_i$; so $\{a,b\} \not\subset B_i \cup B_{\sigma(i)}$. We may assume that $a \in B_i$, $b \in B_{\sigma^2(i)}$. Let $x, y \in A$. If $\{x,y\} \subseteq B_i \cup B_{\sigma(i)}$ or $\{x,y\} \subseteq B_{\sigma^2(i)}$ then $(x,y) \in m_i$. If $\{x,y\} \not\subset B_i \cup B_{\sigma(i)}$ and $\{x,y\} \not\subset B_{\sigma^2(i)}$, we may assume that $x \in B_i \cup B_{\sigma(i)}$ and $y \in B_{\sigma^2(i)}$; so, $(x,a),(b,y) \in m_i \subseteq \theta$ implies $(x,y) \in \theta$. So, $\theta = A \times A$. Finally, let $\theta \in Con\overline{A}$ which $\theta \not\subset m_i$ for all $i \in \{1,2,3\}$. There are $(a,b) \in \theta$, $(c,d) \in \theta$, $(p,q) \in \theta$, $(a,b) \notin m_i$, $(c,d) \notin m_{\sigma(i)}$, $(p,q) \notin m_{\sigma^2(i)}$ for $i \in \{1,2,3\}$. Hence, $a \in B_i \cup B_{\sigma(i)}$, $b \in B_{\sigma^2(i)}$, $c \in B_{\sigma(i)} \cup B_{\sigma^2(i)}$, $d \in B_i$, $p \in B_{\sigma^2(i)} \cup B_i$ and $q \in B_{\sigma(i)}$. If $a \in B_i$, the cyclically of f and $(a,b) \in \theta$ implies that $(x,y) \in \theta$ for all $x, y \in B_i \cup B_{\sigma^2(i)}$. But $(p,q) \in \theta$, we have either $(s,t) \in \theta$ for all $s,t \in B_{\sigma(i)} \cup B_{\sigma^2(i)}$ or $(s,t) \in \theta$ for all $s,t \in B_i \cup B_{\sigma(i)}$. In any cases, $(s,t) \in \theta$ for all $s,t \in A$. Hence, $\theta = A \times A$. We can also prove that $\theta = A \times A$ similarly if $a \in B_{\sigma(i)}$.

Clearly, $m_i \vee m_{\sigma(i)} = A \times A$, for all $i \in \{1,2,3\}$. Let m be the greatest element of the sublattice $C := \bigcap_{i=1}^{3} C_i$. It is clear that m is the greatest lower bound of $\{m_1, m_2, m_3\}$. So, $\{m, m_1, m_2, m_3, A \times A\}$ is a sublattice of $Con\overline{A}$ which is isomorphic to M_3. Therefore, $Con\overline{A}$ is not distributive.

Note that : if $\theta, \phi \in Con\overline{A}$ with $\phi \subseteq \theta$ then $\theta, \phi \in C_i$ for some $i \in \{1,2,3\}$. We will now show that $Con\overline{A}$ has no sublattice which is isomorphic to N_5. Let $\theta, \phi, \varphi \in Con\overline{A}$ such that $\phi \subseteq \theta$, $\varphi \mid \theta$ and $\varphi \mid \phi$. Then, $\theta, \phi \in C_i$ for some $1 \leq i \leq 3$. If $\varphi \in C_i$, the distributivity of C_i implies that $\phi \wedge \varphi \neq \theta \wedge \varphi$ and $\phi \vee \varphi \neq \theta \vee \varphi$. If $\varphi \in C_j$ for some $1 \leq j \neq i \leq 3$ then $\varphi \in C$ implies that $\phi \wedge \varphi \neq \theta \wedge \varphi$ and $\phi \vee \varphi \neq \theta \vee \varphi$; and if $\varphi \in C_j \setminus C$ then $\theta \in C_i \setminus C$ and $\phi \in C$ imply that $\theta \vee \varphi = A \times A$ and $\phi \vee \varphi = m_j$; so, $\phi \vee \varphi \subset \theta \vee \varphi$; and $\theta, \phi \in C_i \setminus C$ impliesthat $\theta = \theta_{B_i \cup B_{\sigma(i)}} \cup \theta_{B_{\sigma^2(i)}}$ and $\varphi = \varphi_{B_j \cup B_{\sigma(j)}} \cup \varphi_{B_{\sigma^2(j)}}$ which also imply that $\theta \wedge \varphi = (\theta_{B_i \cup B_{\sigma(i)}} \cup \theta_{B_{\sigma^2(i)}}) \cap (\varphi_{B_j \cup B_{\sigma(j)}} \cup \varphi_{B_{\sigma^2(j)}})$ $\subset (\phi_{B_i \cup B_{\sigma(i)}} \cup \phi_{B_{\sigma^2(i)}}) \cap (\varphi_{B_j \cup B_{\sigma(j)}} \cup \varphi_{B_{\sigma^2(j)}})$ $= \phi \wedge \varphi$. Therefore, $Con\overline{A}$ is modular. □

Corollary 1 If a symmetric algebra $\overline{A}$ is congruence-modular then there exist co-atoms m_1, m_2 and m_3 of $Con\overline{A}$ which satisfy the following conditions:

(i) for each $i \in \{1,2,3\}$, $\downarrow m_i$ is one of the lattices P, $P \oplus \underline{1}$ or $(P \oplus \underline{1}) \times Q$ where P and Q are product of chains.

(ii) the set $\{m, m_1, m_2, m_3, A \times A\}$ forms a M_3 − sublattice of $Con\overline{A}$ where m is the greatest element of $\bigcap_{i=1}^{3} \downarrow m_i$.

Definition 1 A lattice L with the greatest element 1 is said to be a M_3-head lattice if

(i) L contains exactly three co-atoms m_1, m_2 and m_3 where $\downarrow m_i$ satisfies Condition (i) of Corollary 1 for all $i \in \{1,2,3\}$, and

(ii) The set $\{m, m_1, m_2, m_3, 1\}$ forms a M_3 − sublattice of L where m is the greatest element of $\bigcap_{i=1}^{3} \downarrow m_i$.

We have the following characterizations.

Theorem 1 The followings are equivalence for a symmetric algebra $\overline{A}$.

(i) $\overline{A}$ is congruence modular,

(ii) Conditions (i), (ii) or (iii) of Proposition 3 are satisfied,

(iii) $Con\overline{A}$ is either a product of chains, a linear sum of a product of chains with one element chain or a M_3-head lattice.

All lattices which are congruence lattices of modular near-symmetric algebras.

Let f be a unary operation on a finite set A with $\lambda(f) = 1$ which we will call $\overline{A} = (A; f)$, a near-symmetric algebra. The first proposition proves a characterizations of f.

Proposition 5. The followings are equivalent for a near-symmetric algebra $\overline{A} = (A; f)$.

(i) $\lambda(f) = 1$.

(ii) There is a $\phi \neq B \subset A$ such that $B \cap \operatorname{Im} f = \phi$ and $f\big|_{A \setminus B}$ is a permutation.

(iii) $\operatorname{Im} f \subset A$ and $f\big|_{\operatorname{Im} f}$ is a permutation.

(iv) $\operatorname{Im} f \subset A$ and $B \cap \operatorname{Im} f$ is a one-element set for all $B \in A/\ker f$.

Proof. Let $\lambda(f) = 1$. Then $\operatorname{Im} f \subset A$; so, there is a $\phi \neq B \subset A$ such that $A = B \cup \operatorname{Im} f$ is a disjoint union and $f\big|_{\operatorname{Im} f^{\lambda(f)}}$ is a permutation.

If (ii) holds, then $\operatorname{Im} f \subseteq A \setminus B \subset A$. Since $f\big|_{A\setminus B}$ is a permutation and $\operatorname{Im} f\big|_{A\setminus B} \subseteq \operatorname{Im} f$, we have $A \setminus B \subseteq \operatorname{Im} f$. So, $A \setminus B = \operatorname{Im} f$. Hence, $f\big|_{\operatorname{Im} f}$ is a permutation.

Assume that (iii) holds. Let $B \in A/\ker f$. Then $f(B) = \{c\}$ for some $c \in A$. Since $f\big|_{\operatorname{Im} f}$ is a permutation, $B \cap \operatorname{Im} f$ is singleton.

From (iv), we have $f\big|_{\operatorname{Im} f}$ is a permutation; and together with $\operatorname{Im} f \subset A$ imply that $\lambda(f) = 1$. □

We note that $Con\overline{A} = \{\Delta_A, \nabla_A\}$ for all two-elements algebras $\overline{A}$, so we will consider algebras whose cardinalities are more than two.

Proposition 6. Let $\overline{A}$ be a near-symmetric algebra with $|A| \geq 3$.

(i) If $\overline{A}$ is congruence-modular then $A/\ker f$ contains only one block whose cardinality more than one.

(ii) If $\overline{A}$ is congruence-distributive then $|\operatorname{Im} f| = |A| - 1$.

Proof. Let $\overline{A}$ be congruence-modular. Proposition 5 tells the existence of the blocks $B_1, B_2, ..., B_s$ whose cardinalities are more than one for some $s \geq 1$ and $B_i \cap \operatorname{Im} f$ for all blocks in $A/\ker f$. Let the permutation $f\big|_{\operatorname{Im} f}$ be a product of disjoint cycles $\alpha_1, \alpha_2, ..., \alpha_r$ for some $r \geq 2$. If $s \geq 2$, then $|B_1| > 1$ and $|B_2| > 1$; hence, $f(B_1) = \{b_1\} \neq \{b_2\} = f(B_2)$; so, there are $u \in B_1$ and $v \in B_2$ such that $u, v \notin \operatorname{Im} f$ and there are $a_1 \neq a_2$ such that $a_i \in B_i \cap \operatorname{Im} f$ for $i \in \{1,2\}$; thus, a_i and b_i are in the same cycle for $i \in \{1,2\}$. Let C be the union of cycles containing $\{b_1, b_2\}$. Then the congruences $\Delta_A \subset \theta(C) \subset \theta(C) \cup \theta(u,v)$, $\ker f$, and $\theta(C) \vee \ker f$ will generate a sublattice which is isomorphic to N_5, a contradiction. So, $s = 1$.

If $\overline{A}$ is congruence-distributive, Part (i) implies that $A/\ker f$ contains exactly one non-singleton block B. Since the least congruence $\theta(x,y)$, $\theta(y,z)$ and $\theta(x,z)$ will generate a M_3-sublattice if x, y, z are distinct in B; so, $|B| = 2$. Hence, $|\operatorname{Im} f| = |A| - 1$. □

Proposition 7. If $\overline{A}$ is a near-symmetric algebra with $|A| \geq 4$ and $|\operatorname{Im} f| = |A| - 1$, then $Con\overline{A} \cong \underline{2} \times Con(\operatorname{Im} f; f)$.

Proof. Let B be the only block of $A/\ker f$ whose $|B| = 2$. Then $f(u) = f(b)$ for all $u \in B \setminus \operatorname{Im} f$ and $b \in B \cap \operatorname{Im} f$. Then $\theta \cup \{(u,u)\} \in Con\overline{A}$ and $\overline{\theta} := \theta \cup \{(x,y) \mid x, y \in [b]_\theta \cup \{u\}\} \in Con\overline{A}$ for all $\theta \in Con(\operatorname{Im} f; f)$. So, the map $g : (1,\theta) \to \overline{\theta}$ and $g : (0,\theta) \to \theta \cup \{(u,u)\}$ is an order-embedding from $\underline{2} \times Con(\operatorname{Im} f; f)$ into $Con\overline{A}$.

Now, let $\theta \in Con\overline{A}$. If $[u]_\theta$ is singleton,

$\varphi = \theta \setminus \{(u,u)\} \in Con(\operatorname{Im} f; f)$ and $g(0,\varphi) = \theta$; and, if $[u]_\theta$ is not singleton, $f(u) = f(b)$ for all $b \in B \cap \operatorname{Im} f$; so, $g(1,\varphi) = \theta$ where φ is the corresponding congruence to the partition of $\operatorname{Im} f$ containing the block of $f(u)$. Therefore, g is an order-isomorphism.

Note that if $\left|\operatorname{Im} f\right| = |A| - 1 \geq 3$, then the results in Remark 2 implies that $Con(\operatorname{Im} f; f)$ is not distributive. So, if $\left|\operatorname{Im} f\right| = |A| - 1 \geq 3$ or $f\big|_{\operatorname{Im} f}$ is an identity, the proof of Proposition 7 shows that $Con\overline{A}$ is not distributive.

Theorem 2. The followings are equivalent for a near-symmetric algebra $\overline{A}$ whose $|A| \geq 4$.

(i) $\overline{A}$ is congruence-distributive.

(ii) $\left|\operatorname{Im} f\right| = |A| - 1$ and $(\operatorname{Im} f; f)$ is congruence-distributive.

(iii) $\left|\operatorname{Im} f\right| = |A| - 1$ and $f\big|_{\operatorname{Im} f}$ is one of (i) or (ii) of Proposition 1.

(iv) $Con\overline{A}$ is either $\underline{2} \times P$ or $\underline{2} \times (P \oplus \underline{1})$ where P is a product of chains.

Proof. $(i) \Rightarrow (ii)$ is clear from Proposition 7 and $(\operatorname{Im} f; f)$ is congruence-distributive. $(ii) \Rightarrow (iii)$ follows from Proposition 1 and the argument after Proposition 7. $(iii) \Rightarrow (iv)$ is clear from Proposition 1, Proposition 4 and Proposition 7. Finally, $(iv) \Rightarrow (i)$ because the lattices $\underline{2} \times P$ and $\underline{2} \times (P \oplus \underline{1})$ are distributive if P is a product of chains. □

Proposition 8. Let $\overline{A}$ be a near-symmetric congruence-modular with $|A| \geq 4$. Then

(i) $\left|\operatorname{Im} f\right| = |A| - 1$ or $\left|\operatorname{Im} f\right| = |A| - 2$,

(ii) if $\left|\operatorname{Im} f\right| = |A| - 1$, then $Con\overline{A} \cong \underline{2} \times Con(\operatorname{Im} f; f)$, and

(iii) if $\left|\operatorname{Im} f\right| = |A| - 2$, then $Con\overline{A} \cong M_3 \times Con(\operatorname{Im} f; f)$.

Proof. Suppose that $\left|\operatorname{Im} f\right| \leq |A| - 3$. There are distinct $a, b, c, d \in A$ with $f(a) = f(b) = f(c) = f(d)$. So, the congruence $\theta(a,b)$, $\theta(a,b) \cup \theta(c,d)$ and $\theta(a,c) \cup \theta(b,d)$ will generate a N_5-sublattice, a contradiction. But, $\left|\operatorname{Im} f\right| \leq |A| - 1$ implies $\left|\operatorname{Im} f\right| = |A| - 1$ or $\left|\operatorname{Im} f\right| = |A| - 2$. One can see that (ii) follows from Proposition 7. We assume that $\left|\operatorname{Im} f\right| = |A| - 2$. Then, $A/\ker f$ contains only one non-singleton block B. If $|B| = 2$ then $\left|\operatorname{Im} f\right| = |A| - 1$, a contradiction. If $|B| \geq 4$, , the proof of Proposition 8 (i) implies a contradiction. Hence, $|B| = 3$.

Let $B = \{a,b,c\}$. Then $f(a) = f(b) = f(c)$ and because $\left|B \cap \operatorname{Im} f\right| = 1$, we may assume that $c \in B \cap \operatorname{Im} f$. Now, the set $\overline{M}_3 = \{\Delta_B, \theta(a,b), \theta(b,c), \theta(a,c)\theta(B)\}$ forms a M_3-sublattice of $Con\overline{A}$. Note that for each $\phi \in Con(B; f\big|_B)$ and $\theta \in Con(\operatorname{Im} f; f)$ the relations $\overline{\phi} = \phi \cup \{(x,x) \mid x \in \operatorname{Im} f\}$ and $\overline{\theta} = \theta \cup \{(x,x) \mid x \in B\}$ are in $Con\overline{A}$. Now, let define $\alpha : M_3 \times Con(\operatorname{Im} f; f) \to Con\overline{A}$ and $\beta : Con\overline{A} \to M_3 \times Con(\operatorname{Im} f; f)$, by $\alpha(\phi,\theta) = \overline{\phi} \vee \overline{\theta}$ for all $(\phi,\theta) \in M_3 \times Con(\operatorname{Im} f; f)$ and $\beta(\theta)\ (\theta\big|_B, \theta\big|_{\operatorname{Im} f})$ for all $\theta \in Con\overline{A}$, respectively. By routine work, one can prove that $\alpha \circ \beta = id_{Con\overline{A}}$ and $\beta \circ \alpha = id_{M_3 \times Con(\mathrm{Im} f; f)}$. Therefore, $Con\overline{A} \cong M_3 \times Con(\mathrm{Im} f; f)$. □

Lemma 1. Let $\overline{A}$ be a near-symmetric algebra whose $|A| \geq 6$ and either $|\mathrm{Im} f| = |A| - 1$ or $|\mathrm{Im} f| = |A| - 2$. If $f|_{\mathrm{Im} f}$ is an identity, $Con\overline{A}$ is not modular. □

We have the following characterizations.

Theorem 3. The followings are equivalent for a near-symmetric algebra $\overline{A}$ whose $|A| \geq 4$.

(i) $\overline{A}$ is congruence-modular.

(ii) $(\mathrm{Im} f; f)$ is congruence-modular and either $|\mathrm{Im} f| = |A| - 1$ or $|\mathrm{Im} f| = |A| - 2$.

(iii) $f|_{\mathrm{Im} f}$ is one of (i) or (ii) or (iii) of Proposition 6 and either $|\mathrm{Im} f| = |A| - 1$ or $|\mathrm{Im} f| = |A| - 2$.

(iv) $Con\overline{A}$ is the lattice $\underline{2} \times P$, $\underline{2} \times (P \oplus \underline{1})$, $\underline{2} \times L$, $M_3 \times P$, $M_3 \times (P \oplus \underline{1})$ or $M_3 \times L$ where P is a product of chains and L is a M_3-head lattice. □

References

Davey, B. A. and Priestley, H. A. (1990) Introduction to Lattices and Order. In *Cambridge Mathematical Textbooks*, New York.

Denecke, K. and Wismath, S. L. (2002) Universal Algebra and Applications in Theoretical Computer Science. In *Chapman & Hall*, New York.

Jakubikova, D. and Kosice (1982) On congruence relations of unary algebras I. *Czechoslovak Mathematical Journal* 32(107): 437 -459.

Jakubikova, D. and Kosice (1983) On congruence relations of unary algebras I. *Czechoslovak Mathematical Journal* 33(108): 448 -466.

McKenzie, R. and Hobby, D. (1998) The structure of finite algebras. *Contemporary Mathematics* vol. 76, Providence, Rhode Island.

Ratanaprasert, C. and Denecke, K. (2008) Unary operations with long pre-periods. *Discrete Mathematics* 308: 4998 – 5005.